Punctured Turbo Code Ensembles

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Abstract — **We analyze the asymptotic performance of punctured turbo codes. The analysis is based on the union bound on the word error probability of maximum likelihood decoding for a punctured turbo code ensembles averaged over all possible puncturing patterns and interleavers. By using special probabilistic puncturing, we prove that, for a given mother turbo code ensemble $[\mathcal{C}]$ with a finite noise threshold $c_0^{[\mathcal{C}]}$, if the asymptotic puncturing rate λ satisfies $\log \lambda < -c_0^{[\mathcal{C}]}$, there exists a finite noise threshold $c_0^{[\mathcal{C}_P]}$ for the punctured turbo code ensemble which is bounded by a function of $c_0^{[\mathcal{C}]}$ and λ. Based on this result, we prove that, on any binary-input memoryless channel whose Bhattacharyya noise distance is greater than $c_0^{[\mathcal{C}_P]}$, the average ML decoding word error probability of the punctured turbo code ensemble approaches zero at least as fast as $n^{-\beta}$, where β is the well known "interleaver gain" exponent. This enables us to answer an important question in the practice of HARQ schemes, namely up to which puncturing rate "good" turbo codes give rise to"good" punctured codes.**

I. Code and Channel Models

A. Turbo Code Ensembles and Random Puncturing

The general structure of a punctured parallel turbo code is shown in Figure 1. The punctured turbo code is obtained by

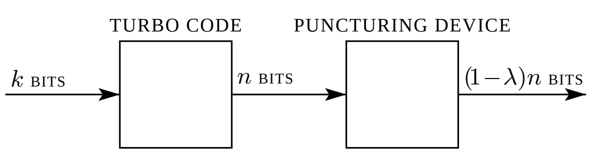

Figure 1: Punctured turbo codes. The puncturing device punctures each bit independently with probability λ. The expected number of remaining bits equals $(1-\lambda)n$

puncturing a mother turbo code. The mother turbo code consists of L pseudorandom interleavers, and L recursive convolutional encoders. There are $k!$ possible choices for each interleaver. Consequently, for a given set of L recursive convolutional encoders, there are $(k!)^L$ different (n,k) turbo codes, corresponding to all different interleavers. We denote this *set* by $\mathcal{C}^{(n)}$. By the mother turbo code *ensemble* $[\mathcal{C}]$, we will mean a *sequence* of turbo code sets $\{\mathcal{C}^{(n)}\}$ with a common rate. Asymptotic properties when $n \to \infty$ of such ensembles were studied in [1].

The puncturing device punctures each codeword symbol independently with probability λ. Therefore, if an (n,k) rate R code is punctured, the expected number of punctured bits equals λn and the resulting code expected rate, as well as the asymptotic rate when $n \to \infty$, equals $R/(1-\lambda)$. For each mother code in the set $\mathcal{C}^{(n)}$, we will have a set of 2^n punctured codes corresponding to the 2^n possible puncturing patterns. Each code in this set appears with the probability determined by the puncturing pattern. The set of all codes obtained by puncturing all codes in the set $\mathcal{C}^{(n)}$ is denoted $\mathcal{C}_P^{(n)}$. By the punctured code turbo code *ensemble* $[\mathcal{C}_P]$, we will mean a *sequence* of randomly punctured turbo code sets $\{\mathcal{C}_P^{(n)}\}$.

B. Performance Measure

We will assume a binary input memoryless channel with output alphabet Ω and transition probabilities $p(y|0)$ and $p(y|1)$, $y \in \Omega$. When (a possibly coded) sequence $\boldsymbol{x} \in \mathcal{C} \subseteq \{0,1\}^n$ has been transmitted, the probability that the maximum likelihood (ML) detector finds sequence $\boldsymbol{x'}$ at Hamming distance d from $\boldsymbol{x}$ more likely can be bounded as follows:

$$P_e(\boldsymbol{x}, \boldsymbol{x'}) \leq \exp\{-d\alpha\},$$

where α is the *Bhattacharyya noise distance* between inputs 0 and 1 defined as

$$\alpha = -\log \sum_{y \in \Omega} \sqrt{p(y|x=0)p(y|x=1)} \qquad (1)$$

if Ω is discrete and

$$\alpha = -\log \int_{\Omega} \sqrt{p(y|x=0)p(y|x=1)}\, dy \qquad (2)$$

if Ω is a measurable subset of $\mathcal{R}$. Thus, for an (n,k) binary linear code C with A_d codewords of weight d, we have the well known *union-Bhattacharyya bound* on the ML decoder word error probability

$$P_W^C \leq \sum_{h=1}^{n} A_d e^{-\alpha d}.$$

In this paper we are concerned with the union bound on the expected average word error probability of punctured turbo code ensembles described above.

II. Weight Enumerators for Punctured Turbo Code Ensembles

For analyzing the expected error rate performance of the punctured codes, we need to find their expected weight enumerators. If a codeword of weight d enters a puncturing device, the codeword at the output will have a weight j with probability $\binom{d}{j}(1-\lambda)^j \lambda^{d-j}$. For a turbo code ensemble $[\mathcal{C}]$, the average number of codewards of weight d is denoted by $\overline{A}_d^{[\mathcal{C}](n)}$. Since on the average $\overline{A}_d^{[\mathcal{C}](n)}$ codewords of weight d enter the puncturing device, the expected number of punctured codewords of weight j is given by

$$\overline{A}_j^{[\mathcal{C}_p](n)} = \sum_{d\geq j} \overline{A}_d^{[\mathcal{C}](n)} \binom{d}{j} \lambda^{d-j}(1-\lambda)^j. \tag{3}$$

The average number of codewords of weight d can be bounded in terms of the ensemble *noise threshold* defined as follows: Let D_n be a sequence of numbers such that

$$D_n \to \infty \quad \text{and} \quad \frac{D_n}{n^\epsilon} \to 0 \;\; \forall \epsilon > 0. \tag{4}$$

Similarly as in [1], we define the *noise threshold* for code ensemble $[\mathcal{C}]$ as

$$c_0^{[\mathcal{C}]} = \limsup_{n\to\infty} \max_{D_n<d\leq n} \frac{\log \overline{A}_d^{[\mathcal{C}](n)}}{d}.$$

It was shown in [1] that for a turbo code ensemble $[\mathcal{C}]$ with the number of interleavers $L \geq 2$, the ensemble noise threshold $c_0^{[\mathcal{C}]}$ is a finite positive number. Therefore, we have

$$\overline{A}_d^{[\mathcal{C}](n)} \leq_n \exp\big(d c_0^{[\mathcal{C}]}\big), \qquad D_n < d \leq n, \tag{5}$$

where $\leq_n$ means that the inequality holds for large enough n.

III. Noise Threshold for the Punctured Turbo Code Ensemble

We define the noise threshold for punctured code ensemble $[\mathcal{C}_P]$ as

$$c_0^{[\mathcal{C}_P]} = \limsup_{n\to\infty} \max_{D_n<j\leq n} \frac{\log \overline{A}_j^{[\mathcal{C}_P](n)}}{j}.$$

To estimate $c_0^{[\mathcal{C}_P]}$, we find a bound on $\overline{A}_j^{[\mathcal{C}_P](n)}$ as follows:

Lemma 1 *Let D_n be a sequence of numbers satisfying (4), and j a weight index $D_n < j \leq n$. If the codeword symbols of a turbo code ensemble $[\mathcal{C}]$ with a finite noise threshold $c_0^{[\mathcal{C}]}$ are punctured with probability λ satisfying $\log\lambda < -c_0^{[\mathcal{C}]}$, then the expected number of punctured codewords of weight j averaged over the ensemble can be bounded as follows:*

$$\overline{A}_j^{[\mathcal{C}_P](n)} \leq_n K(\epsilon) \left[\frac{1-\lambda}{\exp\big(-c_0^{[\mathcal{C}]}\big) - \lambda e^\epsilon}\right]^j \tag{6}$$

for any $\epsilon \in (0\,, -\log\lambda - c_0^{[\mathcal{C}]})$ and $K(\epsilon) = 1/(1-e^{-\epsilon})$.

Proof: By adding some positive terms in (3), we obtain

$$\begin{aligned}\overline{A}_j^{[\mathcal{C}_P](n)} &= \sum_{d\geq j} \overline{A}_d^{[\mathcal{C}](n)} \binom{d}{j} \lambda^{d-j}(1-\lambda)^j \\ &\leq \sum_{d\geq j} \overline{A}_d^{[\mathcal{C}](n)} \sum_{d'=0}^{j} \binom{d}{d'} \lambda^{d-d'}(1-\lambda)^{d'}\end{aligned}$$

We next use (5) to bound $\overline{A}_d^{[\mathcal{C}](n)}$, and the Chernoff bound to bound the probability that d' does not exceed j:

$$\begin{aligned}\overline{A}_j^{[\mathcal{C}_P](n)} &\leq_n \sum_{d\geq j} \exp\{d c_0^{[\mathcal{C}]}\}\, e^{-sj}\, \mathrm{E}[e^{d's}|d] \qquad \forall s<0 \\ &= \sum_{d\geq j} \exp\{d c_0^{[\mathcal{C}]}\}\, e^{-sj} \big((1-\lambda)e^s + \lambda\big)^d \\ &= e^{-sj} \sum_{d\geq j} \exp\big\{d\big[c_0^{[\mathcal{C}]} + \log\big((1-\lambda)e^s+\lambda\big)\big]\big\}.\end{aligned} \tag{7}$$

If $\log\lambda < -c_0^{[\mathcal{C}]}$, then there exists $\epsilon > 0$ such that $\log\lambda < -c_0^{[\mathcal{C}]} - \epsilon$. Hence, we have

$$0 < \frac{\exp\big(-c_0^{[\mathcal{C}]} - \epsilon\big) - \lambda}{1-\lambda} < 1\,. \tag{8}$$

So, we can set

$$e^s = \frac{\exp\big(-c_0^{[\mathcal{C}]} - \epsilon\big) - \lambda}{1-\lambda}\,. \tag{9}$$

Substituting (9) into (7) gives

$$\begin{aligned}\overline{A}_j^{[\mathcal{C}_P](n)} &\leq_n \left[\frac{1-\lambda}{\exp\big(-c_0^{[\mathcal{C}]} - \epsilon\big) - \lambda}\right]^j \sum_{d\geq j} \exp\{-d\epsilon\} \\ &= K(\epsilon) \left[\frac{1-\lambda}{\exp\big(-c_0^{[\mathcal{C}]}\big) - \lambda e^\epsilon}\right]^j,\end{aligned} \tag{10}$$

where $K(\epsilon) = 1/(1-e^{-\epsilon})$. ■

The bound on $[\mathcal{C}_P]$ follows directly from Lemma 1:

Theorem 1 *If the codeword symbols of a turbo code ensemble $[\mathcal{C}]$ with a finite noise threshold $c_0^{[\mathcal{C}]}$ are punctured with probability λ satisfying $\log\lambda < -c_0^{[\mathcal{C}]}$, then the noise threshold for the punctured code ensemble $[\mathcal{C}_P]$ is bounded as follows:*

$$c_0^{[\mathcal{C}_P]} \leq \log\left[\frac{1-\lambda}{\exp\big(-c_0^{[\mathcal{C}]}\big) - \lambda e^\epsilon}\right] < \infty \tag{11}$$

for any $\epsilon \in (0, -\log\lambda - c_0^{[\mathcal{C}]})$.

The expected average number of punctured codewords of weight j can be bounded in terms of the noise threshold:

$$\overline{A}_j^{[\mathcal{C}_P](n)} \leq_n \exp\big(j c_0^{[\mathcal{C}_P]}\big), \qquad D_n < j \leq n. \tag{12}$$

IV. Interleaver Gain for Punctured Turbo Code Ensembles

For a punctured code ensemble $[\mathcal{C}_P]$, by the *union-Bhattacharyya bound*, the average expected ML decoder word error probability can be bounded as follows:

$$\overline{P}_W^{[\mathcal{C}_P](n)} \;\; \leq \;\; \sum_{j\geq 1} \overline{A}_j^{[\mathcal{C}_P](n)} e^{-\alpha j}. \tag{13}$$

Theorem 2 *For a punctured turbo code ensemble $[\mathcal{C}_P]$ with $L \geq 2$ component encoders, a puncturing rate satisfying $\log\lambda < -c_0^{[\mathcal{C}]}$, and a binary-input memoryless channel whose Bhattacharyya distance $\alpha > c_0^{[\mathcal{C}_P]}$, we have*

$$\overline{P}_W^{[\mathcal{C}_P](n)} = O(n^{-L+2+\epsilon}) \quad \text{for any } \epsilon > 0, \tag{14}$$

where $\overline{P}_W^{[\mathcal{C}_p](n)}$ denotes the ensemble ML decoding error probability.

Proof: Let D_n and G_n denote two fixed sequences of integers satisfying the following as $n \to \infty$:

$$\frac{G_n}{n^\epsilon} \to 0 \; \forall \epsilon > 0, \quad \frac{D_n}{G_n} \to 0, \quad \frac{\log n}{D_n} \to 0. \tag{15}$$

For example, $D_n = \log^2 n$, and $G_n = \log^3 n$ will do.

Since $\alpha - c_0^{[\mathcal{C}_P]} > \delta > 0$, by using (12) in the union bound (13), we get

$$\begin{aligned}\overline{P}_W^{[\mathcal{C}_P](n)} &\leq \sum_{j=1}^{D_n} \overline{A}_j^{[\mathcal{C}_P](n)} e^{-\alpha j} + \sum_{j=D_n+1}^{n} \overline{A}_j^{[\mathcal{C}_P](n)} e^{-\alpha j} \\ &\leq_n \sum_{j=1}^{D_n} \overline{A}_j^{[\mathcal{C}_P](n)} + B(\delta) e^{-D_n \delta} \\ &\leq_n \sum_{j=1}^{D_n} \overline{A}_j^{[\mathcal{C}_P](n)} + O(n^{-\frac{D_n \delta}{\log n}}),\end{aligned} \tag{16}$$

where $B(\delta) = e^{-\delta}/(1-e^{-\delta})$. We next need to bound the first term in the above equation:

$$\begin{aligned}\sum_{j=1}^{D_n} \overline{A}_j^{[\mathcal{C}_P](n)} &= \sum_{j=1}^{D_n}\sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \binom{d}{j} (1-\lambda)^j \lambda^{d-j} \\ &= \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \sum_{j=1}^{D_n} \binom{d}{j} (1-\lambda)^j \lambda^{d-j} \\ &= \sum_{d=1}^{G_n} \overline{A}_d^{[\mathcal{C}](n)} + \\ &\quad \sum_{d=G_n+1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \sum_{j=1}^{D_n} \binom{d}{j} (1-\lambda)^j \lambda^{d-j}\end{aligned} \tag{17}$$

From a result of [1], we know that for the turbo code ensemble $[\mathcal{C}]$, if the number of interleavers is $L \geq 2$,

$$\sum_{d=1}^{G_n} \overline{A}_d^{[\mathcal{C}](n)} = O(n^{-L+2+\epsilon}), \;\; \epsilon > 0 \tag{18}$$

The second term in (17) can be bounded by the Chernoff bound in the manner of the proof of Lemma 1, as follows:

$$\sum_{d=G_n+1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \sum_{j=1}^{D_n} \binom{d}{j} (1-\lambda)^j \lambda^{d-j} \leq$$
$$e^{-sD_n} \sum_{d \geq G_n+1} \exp\big\{d\big[c_0^{[\mathcal{C}]} + \log\big((1-\lambda)e^s + \lambda\big)\big]\big\}$$

With $e^s < 1$ ($s < 0$) is again set as in (9), we obtain

$$\begin{aligned}&\sum_{d=G_n+1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \sum_{j=1}^{D_n} \binom{d}{j} (1-\lambda)^j \lambda^{d-j} \leq \\ &\leq_n \left[\frac{1-\lambda}{\exp(-c_0^{[\mathcal{C}]} - \epsilon) - \lambda}\right]^{D_n} \sum_{d \geq G_n+1} \exp\{-d\epsilon\} \\ &= e^{-sD_n} e^{-G_n\epsilon} B(\epsilon) = B(\epsilon)\exp\{-G_n(\epsilon + \frac{D_n}{G_n}s)\} \\ &\leq_n B(\epsilon) e^{-G_n(\epsilon/2)} = O\big(n^{-\frac{G_n\epsilon}{2\log n}}\big).\end{aligned} \tag{19}$$

where $B(\epsilon) = e^{-\epsilon}/(1-e^{-\epsilon})$.

By combining (16), (17), (18), (19), and applying conditions (15), we obtain the desired result (14). ∎

V. An Application

In mobile and satellite packet data transmission, a special coding scheme, known as the *type-II hybrid-ARQ* (HARQ), achieves higher throughput efficiency then ordinary turbo codes by adapting its error correcting code redundancy to fluctuating channel conditions characteristic in these applications. A type-II HARQ protocol implements *incremental redundancy*, which operates as follows. Initially, the information bits are encoded by a "mother" code and a selected number of parity bits are transmitted. If a retransmission is requested, only additional selected parity bits are transmitted. At the receiving end, the additional parity bits together with the previously received parity bits form a codeword of a punctured mother code allowing for an increased in the error correction capacity. This procedure is repeated after each subsequent retransmission request until all the parity bits of the mother code are transmitted. Given the number of parity bits which are at each stage omitted from the mother code (*i.e.*, punctured and not transmitted), their identity is determined by a *puncturing pattern*. The puncturing is done in the so called *rate compatible* way, which ensures that each member of the family of codes uses all the parity bits from higher rate members.

The standard measure of ARQ protocol efficiency is *throughput*, defined as the average number of user data bits accepted at the receiving end in the time required for transmission of a single bit. The throughput of HARQ schemes is strongly affected by the power of the mother code used in the system and the family of codes obtained by puncturing. Thus recently proposed HARQ schemes use powerful turbo codes, and the design of puncturing patterns is an important issue.

Here we study an example $R = 1/3$ turbo code specified by the 1xEV-DV wireless standard as a mother code. The turbo encoder employs $L = 2$ recursive convolutional (RC) encoders with respective rates $R_1 = 1/2$ and $R_2 = 1$ connected in parallel with an "S-random" interleaver. The component code transfer functions are

$$\begin{aligned}G_1(D) &= \left[1, \frac{1+D+D^3}{1+D^2+D^3}\right], \\ G_2(D) &= \left[\frac{1+D+D^3}{1+D^2+D^3}\right].\end{aligned} \tag{20}$$

We generate a family of rate compatible punctured turbo (RCPT) codes with code rates 0.95, 0.9, 0.85, 0.8, 0.75, 0.7, 0.65, 0.6, 0.55, 0.5, 0.45, 0.4, 0.35, and $1/3$ by probabilistic puncturing. Let λ_j for $j = 1, 2, ..., m$ denote the puncturing rates corresponding to the set of code rates and $\lambda_j > \lambda_k$ for $j > k$. We assume that θ_i for $i = 1, 2, \ldots, n$ is an *i.i.d.* random sequence uniformly distributed over $[0, 1)$. If $\theta_i < \lambda_j$, then the i-th codeword symbol is punctured in the j-th punctured code, otherwise it isn't. Note that $\lambda_j > \lambda_k$ for $j > k$ implies that if the i-th codeword symbol is punctured in a low rate code, that it must also be punctured in a high rate code. Thus, we can generate a set of punctured codes satisfying the rate compatibility condition

We determined the throughput and the average FER performance performance of an HARQ scheme based on RCPT codes defined above by simulation. The simulations were done for the standard interleaver lengths of $k = 384$ and 3840, binary antipodal signaling over an AWGN channel for a range of E_s/N_0. The average throughput performance for this HARQ scheme is plotted in Figure 2. We can see that, for a range

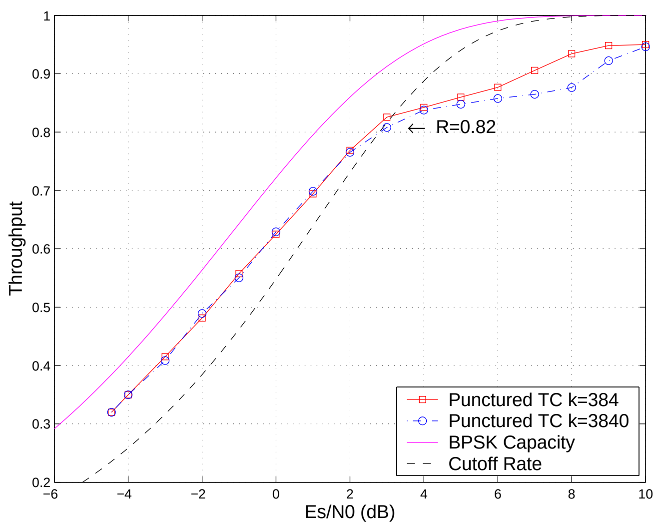

Figure 2: Average throughput performance of punctured turbo code ensembles on an AWGN channel. The mother turbo code with rate=1/3 and interleaver lengths $k = 384$ and 3840 respectively; the punctured code rate=0.95, 0.9, 0.85, 0.8, 0.75, 0.7, 0.65, 0.6, 0.55, 0.5, 0.45, 0.4, 0.35, $1/3$)

of puncturing rates, random puncturing delivered codes whose performance ensured that the throughput be in the region between the cutoff rate and the capacity. After a certain rate the performance of the punctured codes leaves this region. The analysis presented in this paper shows how to estimate this point. We next show that for this particular example, the estimate is in a very good agreement with the numerical value observed in the simulation.

We have computed the average weight enumerators $\overline{A}_d^{[\mathcal{C}](n)}$ of the mother codes set $\mathcal{C}^{(n)}$ over all possible permutations by applying the technique of [2] for code length $n = 1152$ and code rate $R = 1/3$, and obtained

$$c_0^{[\mathcal{C}](n)} = \max_{D_n < d \leq n} \frac{\log \overline{A}_d^{[\mathcal{C}](n)}}{d} = 0.5198.$$

Since $c_0^{[\mathcal{C}]} >_n c_0^{[\mathcal{C}](n)}$, the requirement of Theorem 1 $\log \lambda < -c_0^{[\mathcal{C}]}$ gives

$$\lambda < 0.5946 \text{ and thus } R_p = \frac{R}{1-\lambda} < 0.822. \quad (21)$$

Although Theorem 2 gives only a necessary condition on λ and $[\mathcal{C}_P]$, the simulation results of Fig. 2 show that an abrupt loss in performance occurs roughly for a rate above 0.82.

Figure 3 compares the average FER performance of the punctured turbo code ensembles with interleaver lengths $k = 384$ and 3840. We note that the punctured turbo codes with

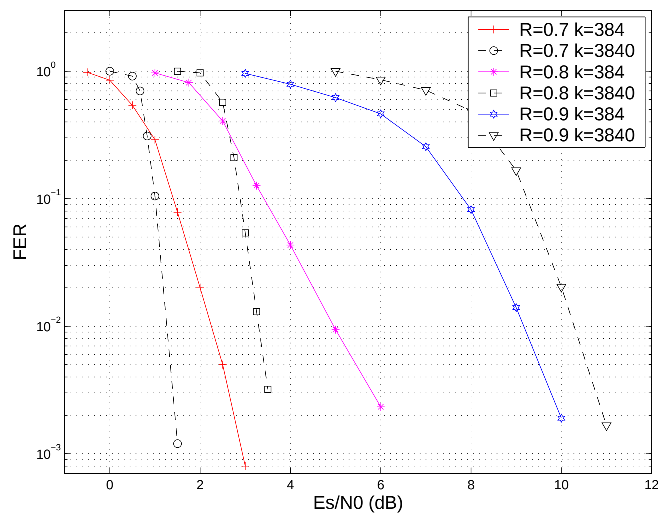

Figure 3: Average FER performance of punctured turbo code ensembles on an AWGN channel. The mother turbo code with rate=$1/3$ and the interleaver lengths $k = 384$ and 3840 respectively; the punctured code rate=0.7, 0.8, 0.9.

code rates $R_p = 0.7$ and 0.8 demonstrate an interleaver gain; whereas in the case of a code rate of $R_p = 0.9$, there is no interleaver gain. The simulation results are again in good agreement with our analysis. Namely, by Theorem 2, if $\log \lambda < -c_0^{[\mathcal{C}]}$, the punctured turbo code has an interleaver gain. In our case, this implies that the punctured turbo codes with rates lower than 0.82 should have an interleaver gain. From Fig. 3, we see that when rates are 0.7 and 0.8, the FER of the $k = 3840$ code drops noticeably faster to 10^{-3} than that of the $k = 384$ code. Interestingly, we also see that when the rate is 0.9, the FER of the $k = 3840$ code drops to 10^{-3} slower than that of the $k = 384$ code.

Acknowledgments

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